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Coupled KdV equations derived from two-layer fluids

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Abstract

Some types of coupled Korteweg de-Vries (KdV) equations are derived from a two-layer fluid system. In the derivation procedure, an unreasonable y-average trick (usually adopted in the literature) is removed. The derived models are classified by means of the Painlevé test. Three types of τ-function and multiple soliton solutions of the models are explicitly given via the exact solutions of the usual KdV equation. It is also discovered that a non-Painlevé integrable coupled KdV system can have multiple soliton solutions.

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1. Introduction

The single-component Korteweg de-Vries (KdV) equation has been widely used in various natural science fields especially in almost all branches of physics. For instance [1], the KdV equation describes, in a general form, competition between weak nonlinearity and weak dispersion, while the nonlinear Schrödinger (NLS) equation describes the same competition for envelope waves (see, for example, the introduction in [2]). Some other integrable equations such as the sine-Gordon (SG) equation, the Kadomtsev–Petviashvily (KP) equation and the so-called three- and four-wave systems are universal as well.

Some kinds of coupled KdV equations have also been introduced in the literature such as one describing two resonantly interacting normal modes of internal-gravity-wave motion in a shallow stratified liquid [3]. In principle, many of other coupled KdV equations are introduced mathematically because of their integrability [4].

In section 2 of this paper, by using a long-wave approximation, we derive some new types of coupled KdV equation systems with some arbitrary parameters from a two-layer fluid model which is used to describe the atmospheric and oceanic phenomena such as the atmospheric blockings, the interactions between the atmosphere and ocean [5], the oceanic circulations and even hurricanes or typhoons [6]. In principle, the atmospheric and oceanic systems should be

(3+1)-dimensional because the density of the fluids is depth or height dependent. To simplify calculations, one usually neglects the nonhomogeneous of the fluids, and then the single-layer (2+1)-dimensional models are used. Starting from the single-layered fluid models, some types of further simplified single-component models such as the KdV, modified KdV, NLS, SG and KP equations can be derived under different types of approximations. In order to get information from the nonhomogeneous of fluids, some types of multiple-layered models have to be utilized. Obviously, the two-layer model is the simplest one among the multiple-layered ones. Beginning with the multiple-layered models, one can derive some types of further simplified multi-component models including the coupled KdV systems.

Once the coupled KdV systems are obtained, an important problem arises as how to solve them. To get more exact solutions, one hopes to pick out the integrable ones. Hence, in section 3, we employ the well-known Painlevé test classification to find out the Painlevé integrable ones for some special types of selections of the parameters.

For some specific types of coupled KdV systems, one can find some types of exact solutions by modifying the solutions of the usual KdV equation. Some concrete examples, particularly, the τ-function and multiple soliton solutions are presented in section 4. The last section contains a short summary and discussion. To simplify the calculations and guarantee the correctness of the results, the computer algebras are used.

2. Coupled KdV equations derived from a two-layer fluid system

It is already known that a great number of integrable models can be derived from fluid dynamics. In this section, we take a two-layer fluid model,

$$q_{1t} + J\{\psi_1, q_1\} + \beta\psi_{1x} = 0, \tag{1}$$

$$q_{2t} + J\{\psi_2, q_2\} + \beta\psi_{2x} = 0, \tag{2}$$

where

$$q_1 = \psi_{1xx} + \psi_{1yy} + F(\psi_2 - \psi_1), \tag{3}$$

$$q_2 = \psi_{2xx} + \psi_{2yy} + F(\psi_1 - \psi_2), \tag{4}$$

and $J\{a, b\} \equiv a_x b_y - b_x a_y$, as a starting point to derive two-component KdV equations by using the multiple-scale approach with a long-wave approximation.

In (1)–(4), F is the weak coupling constant between two layers of the fluid and $\beta = \beta_0(L^2/U)$, $\beta_0 = (2\omega_0/a_0)\cos\phi_0$, where a_0 is the earth's radius, ω_0 is the angular frequency of the earth's rotation and ϕ_0 is the latitude, U is the characteristic velocity scale. The derivation of the dimensionless equations (1) and (2) is based on the characteristic horizontal length scale $L = 10^6$ m and the characteristic horizontal velocity scale $U = 10^{-1}$ m s^{-1} [5].

More specially, when $\beta = 0$, system (3)–(4) is reduced to the usual coupled Euler equation which is suitable to describe the two-layer inviscid fluids. Consequently, all the results obtained in this paper are valid for general two-layer inviscid fluids.

Under the long-wave approximation in the x-direction, in order to derive the KdV-type equations, the stream functions ψ_1 and ψ_2 should have the form

$$\psi_i = \phi_{i0}(y) + \phi_i(\epsilon(x - c_0 t), y, \epsilon^3 t) \equiv \phi_{i0} + \phi_i(X, y, T) \equiv \phi_{i0} + \phi_i, \qquad i = 1, 2, \tag{5}$$

where ϵ is a small parameter. It is reasonably considered that the parameters F and β are in the order ϵ and ϵ^2, respectively, which means that the coupling between two layers is weak and the effect of the rotation of the earth is much smaller. Thus we set

$$F = F_0\epsilon, \qquad \beta = \beta_1\epsilon^2. \tag{6}$$

Now, we expand the shift stream functions ϕ_i $(i = 1, 2)$ as

$$\phi_1 = \epsilon\phi_{11}(X, y, T) + \epsilon^2\phi_{12}(X, y, T) + \epsilon^3\phi_{13}(X, y, T) + \mathrm{O}(\epsilon^4), \tag{7}$$

$$\phi_2 = \epsilon\phi_{21}(X, y, T) + \epsilon^2\phi_{22}(X, y, T) + \epsilon^3\phi_{23}(X, y, T) + \mathrm{O}(\epsilon^4). \tag{8}$$

Substituting (5)–(8) into (1) and (2) yields

$$\begin{aligned}
&[(\phi_{10y} - c)\partial_{yy} - \phi_{10yyy}]\phi_{11X}\epsilon^2 + \{[(\phi_{10y} - c)\partial_{yy} - \phi_{10yyy}]\phi_{12X} + F_0(\phi_{10y} - c)\phi_{21X} \\
&\quad + [F_0(c_0 - \phi_{20y}) + \phi_{11yyy}]\phi_{11X} - \phi_{11y}\phi_{11yyX}\}\epsilon^3 + \{[(\phi_{10y} - c)\partial_{yyX} \\
&\quad - \phi_{10yyy}\partial_X]\phi_{13} - \phi_{12y}\phi_{11yyX} - \phi_{11y}\phi_{12yyX} + (\phi_{10y} - c_0)(F_0\phi_{22} + \phi_{11XX})_X \\
&\quad + \phi_{11yyT} - F_0\phi_{21X}\phi_{11y} + [\phi_{12yyy} + F_0\phi_{21y} + \beta_1]\phi_{11X} \\
&\quad + [F_0(c_0 - \phi_{20y}) + \phi_{11yyy}]\phi_{12X}\}\epsilon^4 + \mathrm{O}(\epsilon^5) = 0,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&[(\phi_{20y} - c)\partial_{yy} - v_{0yyy}]\phi_{21X}\epsilon^2 + \{[(\phi_{20y} - c)\partial_{yy} - \phi_{20yyy}]\phi_{22X} + F_0(\phi_{20y} - c)\phi_{11X} \\
&\quad + [F_0(c_0 - \phi_{10y}) + \phi_{21yyy}]\phi_{21X} - \phi_{21y}\phi_{21yyX}\}\epsilon^3 + \{[(\phi_{20y} - c)\partial_{yyX} \\
&\quad - \phi_{20yyy}\partial_X]\phi_{13} - \phi_{22y}\phi_{21yyX} - \phi_{21y}\phi_{22yyX} + (\phi_{20y} - c_0)(F_0\phi_{12} + \phi_{21XX})_X \\
&\quad + \phi_{21yyT} - F_0\phi_{11X}\phi_{21y} + [\phi_{22yyy} + F_0\phi_{11y} + \beta_1]\phi_{21X} \\
&\quad + [F_0(c_0 - \phi_{10y}) + \phi_{21yyy}]\phi_{22X}\}\epsilon^4 + \mathrm{O}(\epsilon^5) = 0.
\end{aligned} \tag{10}$$

Vanishing the ϵ^2 terms of (9) and (10), we have a special solution

$$\phi_{11} = A_1(X, T)B_1(y) \equiv A_1 B_1, \tag{11}$$

$$\phi_{21} = A_2(X, T)B_2(y) \equiv A_2 B_2, \tag{12}$$

with B_1 and B_2 linked to ϕ_{10} and ϕ_{20} by

$$U_{0yy}B_1 - B_{1y}\phi_{10y} + C_1 = 0, \qquad \phi_{10} = U_0 + c_0 y, \tag{13}$$

and

$$V_{0yy}B_2 - B_{2y}\phi_{20y} + C_2 = 0, \qquad \phi_{20} = V_0 + c_0 y, \tag{14}$$

respectively with arbitrary constants C_1 and C_2.

By using relations (11)–(14), the equations obtained by vanishing the ϵ^3 orders of (9) and (10) and then integrating once with respect to X become

$$2\phi_{10y}(B_1\partial_{yy} - B_{1yy})\phi_{12} + B_1\big[b_{11}A_1^2 - 2F_0(B_1\phi_{20y}A_1 - B_2\phi_{10y}A_2)\big] = 0, \tag{15}$$

$$2\phi_{20y}(B_2\partial_{yy} - B_{2yy})\phi_{22} + B_2\big[b_{21}A_2^2 - 2F_0(B_2\phi_{10y}A_2 - B_1\phi_{20y}A_1)\big] = 0, \tag{16}$$

where the integrating functions have been dropped away and

$$b_{11} \equiv B_1 B_{1yyy} - B_{1y}B_{1yy}, \qquad b_{21} \equiv B_2 B_{2yyy} - B_{2y}B_{2yy}. \tag{17}$$

It is readily verified that

$$\phi_{12} = \big(B_3 A_1^2 + B_0 A_1 + B_4 A_2\big)B_1, \qquad \phi_{22} = \big(B_5 A_2^2 + B_6 A_1 + B_7 A_2\big)B_2, \tag{18}$$

with B_0, B_3, B_4, B_5, B_6 and B_7 being functions of y and determined by

$$B_{0y} = \frac{b_0}{B_1^2}, \qquad b_{0y} = F_0 B_1^2 \frac{g_1}{f_1}, \qquad B_{3y} = \frac{b_3}{B_1^2}, \qquad b_{3y} = -\frac{B_1 b_{11}}{f_1}, \tag{19}$$

$$B_{4y} = \frac{b_4}{B_1^2}, \qquad b_{4y} = -F_0 B_2 B_1, \qquad B_{5y} = \frac{b_5}{B_2^2}, \qquad b_{5y} = -\frac{B_2 b_{21}}{g_1}, \tag{20}$$

$$B_{6y} = \frac{b_6}{B_2^2}, \qquad b_{6y} = -F_0 B_2 B_1, \qquad B_{7y} = \frac{b_7}{B_2^2}, \qquad b_{7y} = F_0 B_2^2 \frac{f_1}{g_1}, \tag{21}$$

$$f_1 = U_{0y}, \qquad g_1 = V_{0y}, \tag{22}$$

solves the third-order equations (15) and (16).

Because of (11), (12) and (18), the fourth order of (9) and (10) becomes

$$\begin{aligned} f_1 \left(\partial_{yy} - B_1^{-1} B_{1yy}\right) \phi_{13X} + B_{1yy} A_{1XT} + f_1 B_1 A_{1XXX} + F_0 (g_1 B_1 B_4 - f_1 B_2 B_7) A_{2X} \\ + 2 f_1 F_0 B_2 B_5 A_2 A_{2X} - (\beta_1 B_1 - F_0 g_1 B_0 B_1 + F_0 f_1 B_2 B_6) A_{1X} \\ + B_4 b_{11} (A_1 A_2)_X + \Bigg[2 b_{11} B_0 - 2 F_0 g_1 B_1 B_3 \\ + \frac{F_0 g_1 B_1}{f_1 B_2} \left(\frac{c_1 B_2}{f_1} - \frac{d_1 B_1}{g_1} - B_2 B_{1y} + B_1 B_{2y} \right) \Bigg] A_1 A_{1X} \\ + \frac{1}{2 f_1^2} \left[b_{11} \left(6 B_3 f_1^2 + 3 f_1 B_{1y} - c_1 \right) - f_1 B_1 b_{11y} \right] A_1^2 A_{1X} = 0, \end{aligned} \tag{23}$$

and

$$\begin{aligned} g_1 \left(\partial_{yy} - B_2^{-1} B_{2yy}\right) \phi_{23X} + B_{2yy} A_{2XT} + g_1 B_2 A_{2XXX} + F_0 (g_1 B_1 B_0 - f_1 B_2 B_6) A_{1X} \\ + 2 g_1 F_0 B_1 B_3 A_1 A_{1X} - (\beta_1 B_2 + F_0 f_1 B_7 B_2 - F_0 g_1 B_1 B_4) A_{2X} \\ + B_6 b_{21} (A_1 A_2)_X + \Bigg[2 b_{21} B_7 - 2 F_0 f_1 B_2 B_5 + \frac{F_0 f_1 B_2}{g_1 B_1} \\ \times \left(\frac{d_1 B_1}{g_1} - \frac{c_1 B_2}{f_1} - B_1 B_{2y} + B_2 B_{1y} \right) \Bigg] A_2 A_{2X} \\ + \frac{1}{2 g_1^2} \left[b_{21} \left(6 B_5 g_1^2 + 3 g_1 B_{2y} - d_1 \right) - g_1 B_2 b_{21y} \right] A_2^2 A_{2X} = 0. \end{aligned} \tag{24}$$

In the usual process of solving (23) and (24) type equations, especially in the atmospheric and oceanic dynamics, one would take ϕ_{13} and ϕ_{23} as zero. However, fixing ϕ_{13} and ϕ_{23} as zero may result in a non-consistent problem because A_1 and A_2 are only the functions of X and T while the coefficients of (23) and (24) are explicitly y-dependent. In general, equations (23) and (24) are not consistent except that all the y-dependent coefficients are proportional to each other up to a constant level. Nonetheless, the detailed analysis of equations (23) and (24) with $\phi_{13} = \phi_{23} = 0$ reveals that it is rather impossible to select ten functions $B_0, B_1, \ldots, B_7, U_0$ and V_0 to be proportional to each other (14 conditions) and satisfy equations (13), (14) and (19)–(21) (22 equations in all!) at the same time. To avoid this kind of inconsistency, in the traditional literature, an unreasonable and unclear procedure is usually made, i.e., taking a y average by integrating the inconsistent equations with respect to the variable y from y_1 to y_2.

Nevertheless, it is possible to get some consistent and significant solutions from (23) and (24) by taking nonzero ϕ_{13} and ϕ_{23}. In this paper, we only give out a possible selection of ϕ_{13} and ϕ_{23} to derive coupled KdV-type equations for the quantities A_1 and A_2.

It is straightforward to verify that if

$$\phi_{13} = r_1 \int A_{1X} A_2 \,\mathrm{dX} + r_2 A_1^3 + r_3 A_1^2 + r_4 A_1 + r_5 A_1 A_2 + r_6 A_2^2 + r_7 A_2 + r_8 A_{1XX}, \tag{25}$$

$$\phi_{23} = s_1 \int A_{1X} A_2 \,\mathrm{dX} + s_2 A_2^3 + s_3 A_2^2 + s_4 A_2 + s_5 A_1 A_2 + s_6 A_1^2 + s_7 A_1 + s_8 A_{2XX}, \tag{26}$$

with

$$r_i = B_1 \int^y \frac{1}{B_1(y'')^2} \int^{y''} R_i(y')\,\mathrm{d}y'\,\mathrm{d}y'',$$

$$s_i = B_2 \int^y \frac{1}{B_2(y'')^2} \int^{y''} S_i(y')\,\mathrm{d}y'\,\mathrm{d}y'', \qquad i = 1, 2, \ldots, 8,$$

$$R_1 = -\frac{\alpha_1 B_1 B_{1yy}}{f_1}, \qquad R_2 = \frac{B_1}{6f_1^3}\left[B_1 f_1 b_{11y} + b_{11}\left(c_1 - 3f_1 B_{1y} - 6B_3 f_1^3\right)\right],$$

$$R_3 = \frac{B_1^2 F_0 g_1}{2f_1}\left(2B_3 + \frac{B_2 B_{1y} - B_1 B_{2y}}{B_2 f_1} - \frac{c_1}{f_1^2}\right) - \frac{B_1}{f_1}(\alpha_5 B_{1yy} + b_{11} B_0) + \frac{F_0 d_1 B_1^3}{f_1^2 B_2},$$

$$R_4 = -\frac{B_1}{f_1}(\beta_1 B_1 - F_0 B_0 B_1 g_1 + F_0 f_1 B_2 B_6), \qquad R_5 = -\frac{B_1}{f_1}(\alpha_3 B_{1yy} + b_{11} B_4),$$

$$R_6 = -\frac{B_1}{f_1}[(\alpha_2 - \alpha_5) B_{1yy} + F_0 f_1 B_2 B_5], \qquad R_7 = \frac{F_0 B_1}{f_1}(g_1 B_1 B_4 - f_1 B_2 B_7),$$

$$R_8 = -\frac{B_1}{f_1}(\alpha_4 B_{1yy} + f_1 B_1), \qquad S_8 = -\frac{B_2}{g_1}(\delta_4 B_{2yy} + g_1 B_2),$$

$$S_1 = \frac{\delta_1 B_2 B_{2yy}}{g_1}, \qquad S_2 = \frac{B_2}{6g_1^3}\left[B_2 g_1 b_{21y} + b_{21}\left(d_1 - 3g_1 B_{2y} - 6B_5 g_1^3\right)\right],$$

$$S_3 = \frac{B_2^2 F_0 f_1}{2g_1}\left(2B_5 + \frac{B_1 B_{2y} - B_2 B_{1y}}{g_1 B_1} - \frac{d_1}{g_1^2}\right) + \frac{c_1 F_0 B_2^3}{2g_1^2 B_1} + \frac{B_2}{g_1}(\delta_5 B_{2yy} - b_{21} B_7),$$

$$S_4 = -\frac{B_2}{g_1}(\beta_1 B_2 - F_0 B_7 B_2 f_1 + F_0 g_1 B_1 B_4), \qquad S_5 = \frac{B_2}{g_1}(\delta_3 B_{2yy} - b_{21} B_6),$$

$$S_6 = \frac{B_2}{g_1}[(\delta_2 - \delta_5) B_{1yy} - F_0 g_1 B_1 B_3], \qquad S_7 = \frac{F_0 B_2}{g_1}(g_1 B_1 B_0 - f_1 B_2 B_6)$$

for arbitrary B_1 and B_2, then A_1 and A_2 satisfy the following coupled KdV-type system:

$$A_{1T} + \alpha_1 A_2 A_{1X} + \left(\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 A_{1XX} + \alpha_5 A_1^2\right)_X = 0, \tag{27}$$

$$A_{2T} + \delta_1 A_2 A_{1X} + \left(\delta_2 A_1^2 + \delta_3 A_1 A_2 + \delta_4 A_{2XX} + \delta_5 A_2^2\right)_X = 0, \tag{28}$$

where ten constants $\{\alpha_i, \delta_i, i = 1, 2, 3, 4, 5\}$ are arbitrary.

Now we are confronted with the important question that how to obtain some exact solutions of the coupled KdV-type system (27)–(28). Before giving out some concrete solutions, we try to make a Painlevé classification at first. That means we are going to give some constraints on the parameters $\{\alpha_i, \delta_i, i = 1, 2, 3, 4, 5\}$ such that the solutions of the model are single valued with respect to an arbitrary singular manifold.

3. Painlevé classification of the coupled KdV system

The Painlevé test is one of the best ways to study nonlinear systems. In this section, we take a standard Painlevé test by using Kruskal's simplification and the computer algebra for the coupled KdV system.

To pass the Painlevé test, four steps are required, the leading order analysis, the resonances determination, the test of the primary branch and the test of the secondary branches, respectively.

The leading order analysis for the coupled KdV system (27)–(28) around the arbitrary manifold ϕ ($\phi = X + \psi(T)$ in Kruskal's simplification) shows that there are two possible categories.

Case 1

$$A_1 \sim \frac{u_0}{\phi^2}, \qquad A_2 \sim \frac{v_0}{\phi^2}. \tag{29}$$

In this case, the parameters $\{\alpha_i, \delta_i\}$ and $\{u_0, v_0\}$ are related by

$$2\alpha_5 u_0^2 + 2\alpha_2 v_0^2 + (2\alpha_3 + \alpha_1)u_0 v_0 + 12\alpha_4 u_0 = 0, \tag{30a}$$

$$2\delta_5 v_0^2 + 2\delta_2 u_0^2 + (2\delta_3 + \delta_1)u_0 v_0 + 12\delta_4 v_0 = 0. \tag{30b}$$

Case 2

$$A_1 \sim \frac{u_0}{\phi^2}, \qquad A_2 \sim \frac{v_0}{\phi} \tag{31}$$

or equivalently

$$A_1 \sim \frac{u_0}{\phi}, \qquad A_2 \sim \frac{v_0}{\phi^2} \tag{31$'$}$$

which will not be considered due to the exchange symmetry $\{A_1, A_2, \alpha_i, \delta_i\} \leftrightarrow \{A_2, A_1, \delta_i, \alpha_i\}$ for the coupled KdV system (27)–(28).

Case (31) appears only for

$$\delta_2 = 0, \qquad \delta_4 = \frac{\alpha_4}{\alpha_5}(2\delta_1 + 3\delta_3), \qquad u_0 = -6\frac{\alpha_4}{\alpha_5}. \tag{32}$$

From the resonance analysis for the first case (29), we know that the resonant points are located at

$$-1, 4, 6, j_1, j_2, j_3 = 9 - j_1 - j_2, \tag{33}$$

where j_1, j_2 and j_3 are three roots of

$$\begin{aligned} &u_0 v_0 \delta_4 \alpha_4 (j-9) j^2 + \big[\alpha_4\big(14\delta_4 v_0 u_0 - u_0^2(v_0\delta_1 + \delta_3 v_0 + 2\delta_2 u_0)\big) - v_0^2\delta_4(2\alpha_2 v_0 + u_0\alpha_3)\big] j \\ &\quad + \alpha_4\big(24\delta_4 v_0 u_0 + 2u_0^2(4\delta_2 u_0 + v_0\delta_1 + 2\delta_3 v_0)\big) \\ &\quad + 2v_0^2\delta_4(u_0\alpha_1 + 2u_0\alpha_3 + 4\alpha_2 v_0) = 0 \end{aligned} \tag{34}$$

for the variable j. Apart from the equivalent decoupled case that both A_1 and A_2 satisfy the completely decoupled KdV equations, the positive integer conditions for the resonant points lead to the following ten nonequivalent subcases: (i) $j_1 = j_2 = 0, j_3 = 9$, (ii) $j_1 = 0, j_2 = 1, j_3 = 8$, (iii) $j_1 = 0, j_2 = 2, j_3 = 7$, (iv) $j_1 = 0, j_2 = 3, j_3 = 6$, (v) $j_1 = 0, j_2 = 4, j_3 = 5$, (vi) $j_1 = j_2 = 1, j_3 = 7$, (vii) $j_1 = 1, j_2 = 2, j_3 = 6$, (viii) $j_1 = 1, j_2 = 3, j_3 = 5$, (ix) $j_1 = j_2 = 2, j_3 = 5$ and (x) $j_1 = 2, j_2 = 3, j_3 = 4$.

The resonance analysis for the second case (31) shows that the resonances will appear at

$$-1, 0, 4, 6, j_1, j_2 = 6 - j_1, \tag{35}$$

where j_1 and j_2 are two solutions of

$$j(2\delta_1 + 3\delta_3)(j-6) + 27\delta_3 + 22\delta_1 = 0 \tag{36}$$

for the variable j. It is clear that the positive integer conditions for the resonance points bring out four nonequivalent subcases: (a) $j_1 = 0, j_2 = 6$, (b) $j_1 = 1, j_2 = 5$, (c) $j_1 = 2, j_2 = 4$ and (d) $j_1 = 3, j_2 = 3$.

Checking all the resonance conditions for subcases (i)–(x) and (a)–(d) yields the possible Painlevé integrable models under some constraints for the parameters α_i and δ_i. For instance, for case (vii), $j = 1, 2$ and 6 are the solutions of (34) only for the following two conditions

$$\alpha_4 u_0^2(\delta_1 v_0 + 2u_0\delta_2 + v_0\delta_3) + v_0\delta_4\big(2\alpha_2 v_0^2 + \alpha_3 u_0 v_0 + 6\alpha_4 u_0\big) = 0, \tag{37a}$$

$$\alpha_4 u_0^2(\delta_1 v_0 + 4u_0\delta_2 + 2v_0\delta_3) + 2\delta_4 v_0\big[4\alpha_2 v_0^2 + (2a_3 + a_1)u_0 v_0 + 18\alpha_4 u_0\big] = 0 \tag{37b}$$

are satisfied. Four conditions (30) and (37) with $\alpha_4 \neq 0$ (which is a requirement for the resulting equation belonging to KdV type) can be simplified to

$$\alpha_5 = -\frac{6\alpha_4}{u_0} - \frac{\alpha_2 v_0^2}{u_0^2} - \frac{(\alpha_1 + 2\alpha_3)v_0}{2u_0}, \qquad \delta_5 = \delta_4\left(\frac{\alpha_3}{2\alpha_4} - \frac{3}{v_0} + \frac{a_2 v_0}{a_4 u_0}\right) - \frac{u_0\delta_3}{2v_0},$$
$$\delta_1 = \frac{\delta_4(6\alpha_4 + \alpha_1 v_0)}{\alpha_4 u_0}, \qquad \delta_2 = -\frac{\delta_4 v_0}{\alpha_4 u_0^2}\left(6a_4 + \frac{v_0}{2}(a_1 + a_3) + \frac{\alpha_2 v_0^2}{u_0}\right) - \frac{\delta_3 v_0}{2u_0}. \tag{38}$$

Now substituting the expansion

$$A_1 = \sum_{i=0}^{\infty} u_j \phi^{j-2}, \qquad A_2 = \sum_{i=0}^{\infty} v_j \phi^{j-2} \tag{39}$$

with (38), where $\phi = X + \psi$ (ψ, u_j and v_j are the functions of T) into the general equation system (27)–(28) and vanishing the coefficients of ϕ^j for $j = -4, -3, -2, -1, 0, 1$ and 2 produces the determining equations of the expansion coefficients $\{u_j, v_j, j = 1, 2, \ldots, 6\}$. Solving these equations one by one, some further consistent conditions have to be inserted to guarantee the compatibility conditions at the resonances $j = 1, 2, 4, 6, 6$.

The final parameter constraints read

$$\delta_1 = -\frac{\alpha_1\alpha_3}{2\alpha_2}, \qquad \delta_2 = \frac{(\alpha_1 - \alpha_3)\alpha_3^2}{8\alpha_2^2}, \qquad \delta_3 = \frac{\alpha_3(2\alpha_1 - \alpha_3)}{2\alpha_2},$$
$$\delta_4 = \alpha_4, \qquad \delta_5 = \alpha_1 - \frac{\alpha_3}{2}, \qquad \alpha_5 = \frac{\alpha_3(\alpha_1 + \alpha_3)}{4\alpha_2}. \tag{40}$$

Under the parameter constraints (40), all the resonant conditions are identically satisfied such that u_1, u_2, v_4, u_6, v_6 and ψ are all free arbitrary functions while the remaining expansion coefficients are

$$v_1 = -\frac{\alpha_3 u_1}{2\alpha_2}, \qquad v_2 = -\frac{\alpha_2\phi_t + \alpha_1\alpha_2 u_2}{2\alpha_1\alpha_2},$$
$$v_3 = \frac{\alpha_3 u_1 \phi_t}{24\alpha_4\alpha_2}, \qquad u_3 = -\frac{u_1\phi_t}{12\alpha_4}, \qquad u_4 = -\frac{u_{1t}\alpha_1 + 24\alpha_2\alpha_4 v_4}{12\alpha_4(\alpha_1 - \alpha_3)},$$
$$u_5 = \frac{\alpha_1^2 u_{2t} - \alpha_2\phi_{tt}}{12\alpha_4\alpha_1^2} - \frac{u_1\big(\alpha_2\phi_t^2 - \alpha_1^2 u_{1t} + 12\alpha_1^2\alpha_4 u_4\big)}{288\alpha_2\alpha_4^2},$$
$$v_5 = \frac{(\alpha_3 - 2\alpha_1)\phi_{tt}}{24\alpha_4\alpha_1^2} - \frac{\alpha_3 u_{2t}}{24\alpha_2\alpha_4} - \frac{\alpha_3 u_1\big(12\alpha_1^2\alpha_4 u_4 + \alpha_2\phi_t^2 - \alpha_1^2 u_{1t}\big)}{576\alpha_2^2\alpha_4^2}.$$

Finally, making a transformation

$$A_1 = \frac{2\alpha_2}{\alpha_1^2\alpha_3}\big(\alpha_1^2 - 6c\alpha_1\alpha_4 - 18\alpha_4\alpha_3\big)U(x, \alpha_4 t) - \frac{2\alpha_2(\alpha_1 - 6c\alpha_4)}{\alpha_1\alpha_3}V(x, \alpha_4 t), \tag{41}$$

$$A_2 = \frac{18\alpha_3\alpha_4 - 18\alpha_1\alpha_4 - \alpha_1^2}{\alpha_1^2}U(x, \alpha_4 t) + V(x, \alpha_4 t), \tag{42}$$

we arrive at the first type of Painlevé integrable (P-integrable) model.

P-integrable model 1.

$$A_{1T} + \left[A_{1XX} - (c+3)(c+6)A_1^2 - c^2A_2^2\right]_X + 2c[(c+6)A_{1X}A_2 + (c+3)A_1A_{2X}] = 0,$$
$$A_{2T} + \left[A_{2XX} - c(c-3)A_2^2 - (c+3)^2A_1^2\right]_X + 2(c+3)[cA_2A_{1X} + (c-3)A_1A_{2X}] = 0, \qquad (43)$$

where c is an arbitrary constant and $\{U, V, \alpha_4 T\}$ have been redenoted by $\{A_1, A_2, T\}$. For the model system (43) there is only one branch with the resonances located at $\{-1, 1, 2, 4, 6, 6\}$ and all the resonance conditions satisfied identically.

After finishing the similar analysis, we know that there are five Painlevé integrable subcases of the coupled KdV system (27)–(28). Here we just write down the final results for other four cases.

P-integrable model 2.

$$A_{1T} + \left(A_{1XX} + \tfrac{1}{2}(c_2 - c_1 - c_1c_2)A_1^2 + c_1A_1A_2 - \tfrac{1}{2}A_2^2\right)_X = 0,$$
$$A_{2T} + \left(A_{2XX} + \tfrac{1}{2}(c_1 - c_2 - 1)A_2^2 + c_2A_1A_2 - \tfrac{1}{2}c_1c_2A_1^2\right)_X = 0, \qquad (44)$$

where c_1 and c_2 are the arbitrary constants. For this type of coupled KdV system (44), there are three branches with the resonances located at $\{-1, 2, 3, 4, 4, 6\}$, $\{-1, 2, 3, 4, 4, 6\}$ and $\{-1, -1, 4, 4, 6, 6\}$, respectively, while all the resonance conditions are identically satisfied.

P-integrable model 3.

$$A_{1T} + \left(A_{1XX} + A_1^2 + A_1A_2\right)_X = 0, \qquad A_{2T} + \left(A_{2XX} + A_2^2 + A_1A_2\right)_X = 0. \qquad (45)$$

In this case, the resonance points are $\{-1, 0, 4, 4, 5, 6\}$.

P-integrable model 4.

$$A_{1T} + [A_{1XX} + (A_1 + A_2)^2]_X = 0, \qquad A_{2T} + [A_{2XX} + (A_1 + A_2)^2]_X = 0. \qquad (46)$$

This case is corresponding to the resonances located at $\{-1, 2, 3, 4, 4, 6\}$.

P-integrable model 5.

$$A_{1T} + \left(A_{1XX} + A_1^2\right)_X + 2A_2A_{1X} = 0, \qquad A_{2T} + \left(A_{2XX} + A_2^2\right)_X + 2A_1A_{2X} = 0. \qquad (47)$$

Now the resonances are situated at $\{-1, 0, 2, 4, 6, 7\}$.

Though it is tedious to figure out the P-integrable models (43)–(47) from the general model (27)–(28), to check the Painlevé property of (43)–(47) is quite easy by means of any version of the P-test method such as the Weiss–Tabor–Carnevale approach, Kruskal's simplification [7], Conte's invariant method [8] and Lou's extended approach [9]. Actually, to verify the Painlevé property of any one model of (43)–(47), nothing needs to do but press an 'enter' key in the environment of any existing algebraic programmes, say, 'P-test' by Xu and Li [10], although all the known existing algebraic programmes including [10] fail to directly figure out the P-integrable models from (27)–(28).

4. Exact solutions

In this section, we study some types of exact solutions for the general coupled KdV system (27)–(28) and some special types of P-integrable models.

4.1. Travelling periodic and solitary wave solutions of the general coupled KdV system (27)–(28)

In [11], it is pointed out that some special types of exact solutions, including the travelling wave solutions, of various nonlinear systems can be obtained via the deformation and mapping approach from the solutions of the cubic nonlinear Klein–Gordon equation (or namely, ϕ^4 model). It is quite easy to see that some types of travelling wave solutions of the coupled KdV system (27)–(28) can also be obtained by some suitable deformation relations from the travelling wave solutions of the ϕ^4 model.

For the travelling wave solutions of the coupled KdV system (27)–(28),

$$A_1 = A_1(\xi) \equiv A_1(kX - kcT), \qquad A_2 = A_2(\xi), \tag{48}$$

we have

$$\alpha_1 A_{1\xi} A_2 + \left(\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 k^2 A_{1\xi\xi} + \alpha_5 A_1^2 - cA_1\right)_\xi = 0, \tag{49}$$

$$\delta_1 A_{1\xi} A_2 + \left(\delta_2 A_1^2 + \delta_3 A_1 A_2 + \delta_4 k^2 A_{2\xi\xi} + \delta_5 A_2^2 - cA_2\right)_\xi = 0. \tag{50}$$

To map the travelling waves of the cubic nonlinear Klein–Gordon equation to those of the coupled KdV system, one can use different mapping relations such as the polynomial forms [11], rational forms [12] and/or more complicated derivative-dependent forms [13]. However, here we just give a simple polynomial deformation relation

$$A_1 = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2, \qquad A_2 = b_0 + b_1\phi(\xi) + ba_2\phi(\xi)^2, \tag{51}$$

where $\phi(\xi)$ is a travelling wave solution of the cubic nonlinear Klein–Gordon equation, i.e., ϕ satisfies

$$\phi_\xi^2 = \mu\phi^2 + \tfrac{1}{2}\lambda\phi^4 + C. \tag{52}$$

It is not very difficult to find that $\{A_1, A_2\}$ given by (51) with (52) is a solution of the coupled KdV system (27)–(28) if and only if the 11 solution parameters $a_0, a_1, a_2, b_0, b_1, b, \mu, \lambda, C, k, c$ and 10 model parameters α_i, δ_i $(i = 1, 2, \ldots, 5)$ satisfy the following eight constraints:

$$\begin{aligned}
&(2\alpha_1 + 3\alpha_3 + 6\alpha_2 b)a_2 b_1 + [a_2(3\alpha_3 b + 6\alpha_5 + \alpha_1 b) + 3k^2\alpha_4\lambda]a_1 = 0,\\
&a_0 a_2(2\alpha_3 b + 4\alpha_5) + 2\alpha_2 b_1^2 + a_1 b_1(\alpha_1 + 2\alpha_3) + 2\alpha_5 a_1^2\\
&\qquad + a_2[8k^2\alpha_4\mu - 2c + (2\alpha_3 + 4\alpha_2 b + 2\alpha_1)b_0] = 0,\\
&a_0(2\alpha_5 a_1 + b_1\alpha_3) + 2\alpha_2 b_0 b_1 + a_1[k^2\alpha_4\mu - c + (\alpha_1 + \alpha_3)b_0] = 0,\\
&a_0(b_1\delta_3 + 2\delta_2 a_1) + b_1(k^2\delta_4\mu + 2b_0\delta_5 - c) + a_1 b_0(\delta_1 + \delta_3) = 0,\\
&a_2(4\delta_5 b^2 + 2\delta_1 b + 4\delta_2 + 4\delta_3 b) + 12k^2\delta_4 b\lambda = 0,\\
&a_0 a_2(4\delta_2 + 2\delta_3 b) + 2\delta_5 b_1^2 + a_1 b_1(2\delta_3 + \delta_1) + 2\delta_2 a_1^2\\
&\qquad + a_2[8k^2\delta_4 b\mu - 2cb + b_0(4\delta_5 b + 2\delta_1 + 2\delta_3)] = 0,\\
&a_2(4\alpha_3 b + 2\alpha_1 b + 4\alpha_2 b^2 + 4\alpha_5) + 12k^2\alpha_4\lambda = 0,\\
&b_1[a_2(6\delta_5 b + 2\delta_1 + 3\delta_3) + 3k^2\delta_4\lambda] + a_1 a_2(3\delta_3 b + \delta_1 b + 6\delta_2) = 0.
\end{aligned} \tag{53}$$

Obviously, the algebraic equation system (53) may possess a great number of solutions. Here we just write down a most important and simplest solution when

$$\delta_4 = \alpha_4, \tag{54}$$

and

$$a_0 = a_1 = b_0 = b_1 = 0, \qquad c = 4k^2\mu\alpha_4, \qquad a_2 = -\frac{6k^2\lambda\alpha_4}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2}, \tag{55}$$

while b is linked to the model parameters by a cubic algebraic equation

$$\delta_2 + \left(\delta_3 - \alpha_5 + \tfrac{1}{2}\delta_1\right)b + \left(\delta_5 - \alpha_3 - \tfrac{1}{2}\alpha_1\right)b - \alpha_2 b^3 = 0. \tag{56}$$

More concretely, if we take $\phi(\xi)$ as the Jacobi elliptic conoid function

$$\phi = \mathrm{cn}(\xi, m)$$

which is a special solution of the ϕ^4 model with the parameters

$$\mu = 2m^2 - 1, \qquad \lambda = -2m^2, \qquad C = 1 - m^2,$$

then we obtain a simple periodic wave solution for the coupled KdV equation (27)–(28) with (54),

$$\begin{aligned} A_1 &= \frac{12k^2m^2\alpha_4}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2}\,\mathrm{cn}^2(kX - 4k^3(2m^2-1)\alpha_4 T, m),\\ A_2 &= \frac{12k^2m^2\alpha_4 b}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2}\,\mathrm{cn}^2(kX - 4k^3(2m^2-1)\alpha_4 T, m), \end{aligned} \tag{57}$$

where b is a solution of (56). Furthermore, when $m = 1$, the periodic solution (57) becomes a simple solitary wave solution

$$\begin{aligned} A_1 &= \frac{12k^2\alpha_4}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2}\,\mathrm{sech}^2(kX - 4k^3\alpha_4 T),\\ A_2 &= \frac{12k^2\alpha_4 b}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2}\,\mathrm{sech}^2(kX - 4k^3\alpha_4 T). \end{aligned} \tag{58}$$

4.2. τ-function solutions and multi-soliton solutions of the coupled KdV system

4.2.1. The first type of τ-function and multi-soliton solutions related to the KdV reductions. It is straightforward to verify that for the coupled KdV equation system (27)–(28) with (54), one can find at least one type of multiple soliton solutions because there is a simple KdV reduction

$$A_{1T} + \alpha_4 A_{1XXX} + (a\alpha_1 + 2\alpha_2 a^2 + 2a\alpha_3 + 2\alpha_5)A_1A_{1X} = 0, \qquad A_2 = aA_1, \tag{59}$$

where a is a solution of the algebraic cubic equation

$$2\alpha_2 a^3 + (\alpha_1 + 2\alpha_3 - 2\delta_5)a^2 + (2\alpha_5 - \delta_1 - 2\delta_3)a - 2\delta_2 = 0. \tag{60}$$

In the present case, the coupled KdV equation system (27)–(28) with (54) possesses the following τ-function solutions:

$$A_1 = \frac{A_2}{a} = \frac{12\alpha_4}{a\alpha_1 + 2\alpha_2 a^2 + 2a\alpha_3 + 2\alpha_5}(\ln\tau)_{XX}, \tag{61}$$

where τ is just the usual τ-function. For the multi-soliton solutions, the τ-function reads

$$\begin{aligned} \tau &= 1 + \sum_{i=1}^{N}\pi_i + \sum_{i_1<i_2}^{N} A_{i_1i_2}\pi_{i_1}\pi_{i_2} + \sum_{i_1<i_2<i_3}^{N} A_{i_1i_2i_3}\pi_{i_1}\pi_{i_2}\pi_{i_3} + \cdots + A_{i_1i_2\cdots i_N}\pi_{i_1}\pi_{i_2}\cdots\pi_{i_N},\\ \pi_i &\equiv \exp\left(k_iX - \alpha_4k_i^3T\right), \qquad A_{i_1i_2\cdots i_k} \equiv \prod_{i_a<i_b,a,b=1,2,\ldots,k} A_{i_ai_b}. \end{aligned} \tag{62}$$

It is interesting and worth indicating that there is only one parameter condition (54) for the multiple soliton solutions (59) while the model has been proved to be non-Painlevé integrable. In other words, the existence of multiple soliton solutions is not a sufficient condition of the integrability.

Especially, because there are three real solutions of (60) for the special coupled KdV equation (44), we can obtain three types of multiple soliton solutions $\{u_1, v_1\}$, $\{u_2, v_2\}$ and

$\{u_3, v_3\}$,

$$v_1 = u_1 = \frac{12}{(c_1-1)(c_2-1)}(\ln\tau)_{XX}, \tag{63}$$

$$u_2 = \frac{12}{(c_1-1)(c_1-c_2)}(\ln\tau)_{XX}, \qquad v_2 = c_1 u_2, \tag{64}$$

and

$$u_3 = \frac{12}{(c_2-1)(c_1-c_2)}(\ln\tau)_{XX}, \qquad v_3 = c_2 u_2, \tag{65}$$

with τ given by (62).

4.2.2. The second type of τ-function and multi-soliton solutions of the coupled KdV system. The multi-soliton solutions of the coupled KdV system listed in the last subsection are obtained from its special KdV reduction. In [14], it has been found that even for the non-integrable cases, the coupled nonlinear system may have many more abundant solitary wave structures. So we believe that for the coupled KdV system there may be other types of multiple soliton solutions.

For instance, if the model parameters have the following conditions:

$$\begin{aligned} &\alpha_1\alpha_2(\alpha_1\delta_3 - \delta_1\alpha_3) \neq 0, && \delta_4 = \alpha_4, \\ &\delta_5 = \frac{1}{2}\alpha_3 + \frac{\alpha_2\delta_1}{\alpha_1} - \frac{\alpha_1\delta_3}{2\delta_1}, && \alpha_5 = -\frac{\delta_1\alpha_3}{2\alpha_1} + \frac{1}{2}\delta_3 + \frac{\alpha_1\delta_2}{\delta_1}, \\ &\alpha_3 = -\frac{2\delta_2\alpha_1^2}{\delta_1^2} - \frac{\alpha_1(\delta_1+\delta_3)}{\delta_1} - \frac{2\delta_1\alpha_2}{\alpha_1}, \end{aligned} \tag{66}$$

then we have a new type of multiple soliton solution

$$A_1 = \frac{12\alpha_1\alpha_4}{\alpha_1\delta_3 - \delta_1\alpha_3}(\ln\tau)_{XX} + \frac{\alpha_1}{\delta_1}A_2, \tag{67}$$

where τ is still the τ-function of the usual KdV equation. For the multi-soliton solutions, τ is still given by (62), while A_2 is related to the τ-function by a *linear* equation

$$\begin{aligned} A_{2T} &+ \frac{12\alpha_1\alpha_4(\delta_1\delta_3 + 2\delta_2\alpha_1)}{\delta_1\alpha_3 - \alpha_1\delta_3}A_{2X}(\ln\tau)_{XX} + \frac{144\delta_2\alpha_1^2\alpha_4^2}{(\delta_1\alpha_3-\alpha_1\delta_3)^2}\left[(\ln\tau)_{XX}^2\right]_X \\ &- \frac{12\alpha_1\alpha_4\left(\delta_1\delta_3 + \delta_1^2 + 2\delta_2\alpha_1\right)}{\delta_1\alpha_3 - \alpha_1\delta_3}A_2(\ln\tau)_{XXX} + \alpha_4 A_{2XXX} = 0. \end{aligned} \tag{68}$$

If the third condition (66) is not satisfied, then a *nonlinear* term

$$\left(\delta_1 + \delta_3 + \frac{\delta_1\alpha_3}{\alpha_1} + \frac{2\delta_2\alpha_1}{\delta_1} + \frac{2\delta_1^2\alpha_2}{\alpha_1^2}\right)A_2A_{2X}$$

has to be added to the left-hand side of (68).

Similarly, under the conditions

$$\alpha_3 \neq 0, \qquad \alpha_1 = \delta_1 = 0, \qquad \delta_4 = \alpha_4, \qquad \delta_5 = \frac{\alpha_2(\delta_3 - c_1)^2}{c_1\alpha_3^3} - \frac{1}{2}, \tag{69}$$

$$\alpha_2 = -\frac{\alpha_3^2\left(2c_1\delta_3 + 2\delta_2 - c_1^2\right)}{2c_1(c_1-\delta_3)^2} \tag{70}$$

with c_1 given by

$$c_1 = \alpha_5 \pm \sqrt{\alpha_5^2 - 2\delta_2}, \tag{71}$$

we can obtain the following new type of multiple soliton solutions,

$$A_1 = \frac{12\alpha_4}{c_1}(\ln\tau)_{XX} + \frac{\alpha_3}{\delta_3 - c_1}A_2, \tag{72}$$

where τ is also given by (62) and A_2 is still linked to the τ-function by a *linear* equation

$$A_{2T} + \alpha_4 A_{2XXX} + \frac{12\alpha_4(c_1\delta_3 + 2\delta_2)}{c_1^2}[A_2(\ln\tau)_{XX}]_X + \frac{144\delta_2(\delta_3 - c_1)\alpha_4^2}{\alpha_3 c_1^2}\left[(\ln\tau)_{XX}^2\right]_X = 0. \tag{73}$$

In the same way, if the parameter condition (70) is not satisfied, then we have to add a nonlinear term

$$\left(\frac{2\alpha_2(\delta_3 - c_1)}{\alpha_3} + \frac{\alpha_3\left(2c_1\delta_3 + 2\delta_2 - c_1^2\right)}{c_1(\delta_3 - c_1)}\right)A_2A_{2X}$$

to the left-hand side of (73).

4.2.3. The third type of τ-function and multi-soliton solutions of the coupled KdV system. Actually, in addition to the above types of multiple soliton solutions, there exist other types of soliton solutions. Here is a further simple example for a more specifical model:

$$\begin{aligned} &A_{1T} + aA_{1XXX} + bA_1A_{1X} + bcA_2A_{2X} = 0, \\ &A_{2T} + aA_{2XXX} + bA_1A_{2X} + bA_2A_{1X} = 0. \end{aligned} \tag{74}$$

For this special model, the first type of multiple-soliton solutions has the form

$$A_1 = \frac{6a}{b}(\ln\tau)_{XX}, \qquad A_2 = \pm\frac{1}{\sqrt{c}}A_1, \tag{75}$$

and the second type of multiple soliton solutions is given by

$$A_1 = \frac{12a}{b}(\ln\tau)_{XX} \pm \sqrt{c}A_2, \tag{76}$$

while A_2 determined by

$$A_{2T} + aA_{2XXX} + 12a[A_2(\ln\tau)_{XX}]_X \pm 2b\sqrt{c}A_2A_{2X} = 0, \tag{77}$$

where τ is also the usual τ-function of the KdV equation.

We can also obtain a third type of multiple soliton solutions of (74) as

$$A_1 = \frac{6a}{b}\left[\ln\left(\tau_1^2 + \tau_2^2\right)\right]_{XX}, \tag{78}$$

$$A_2 = \pm\frac{12a}{b\sqrt{-c}}\left(\arctan\frac{\tau_2}{\tau_1}\right)_{XX}, \tag{79}$$

where

$$\tau \equiv \tau_1 + \mathrm{i}\tau_2$$

is just the usual τ-function of the KdV equation but with *complex* parameters, τ_1 and τ_2 are the real and imaginary parts of τ, respectively.

Figures 1 and 2 are two special interaction plots of the two-soliton solution for the coupled KdV system (74) regarding the field A_1 (78) and A_2 (79), respectively, with

$$\tau = 1 + (1+\mathrm{i})\,\mathrm{e}^{k_1X - k_1^3T} + (1+3\mathrm{i})\,\mathrm{e}^{k_2X - k_2^3T} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}(4\mathrm{i}-2)\,\mathrm{e}^{(k_1+k_2)X - (k_1^3+k_2^3)T},$$

$$k_1 = 1, \qquad k_2 = \frac{3}{2} \tag{80}$$

at times $T = -10, -5, 0, 5$ and 10 .

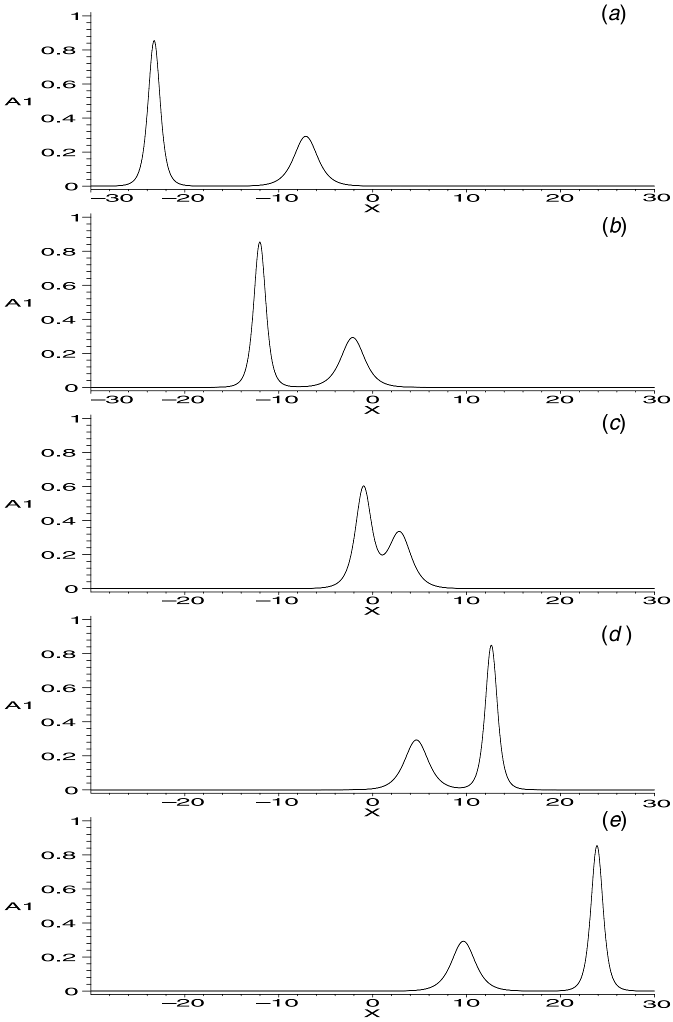

Figure 1. The interaction plots of the two-soliton solution for the field $A1 \equiv A_1$ expressed by (78) and (80) at times (*a*) $T = -10$, (*b*) $T = -5$, (*c*) $T = 0$, (*d*) $T = 5$ and (*e*) $T = 10$, respectively.

5. Summary and discussions

In summary, a general type of the coupled KdV system is derived from the coupled Euler equation system (1)–(2) with ($\beta \neq 0$) and without ($\beta = 0$) the consideration of the earth rotation effects. In the derivation procedure, the frequently used, inconsistent y-average trick in the past literature is removed.

The integrability of the derived KdV system is checked by means of the well-known Weiss–Tabor–Carnevale's Painlevé test procedure. It is found that there are five types of Painlevé integrable subcases for the derived coupled KdV system.

The deformation and mapping method are used to get some types of travelling wave solutions including the conoidal periodic waves and single solitary waves for the general coupled KdV system with (54).

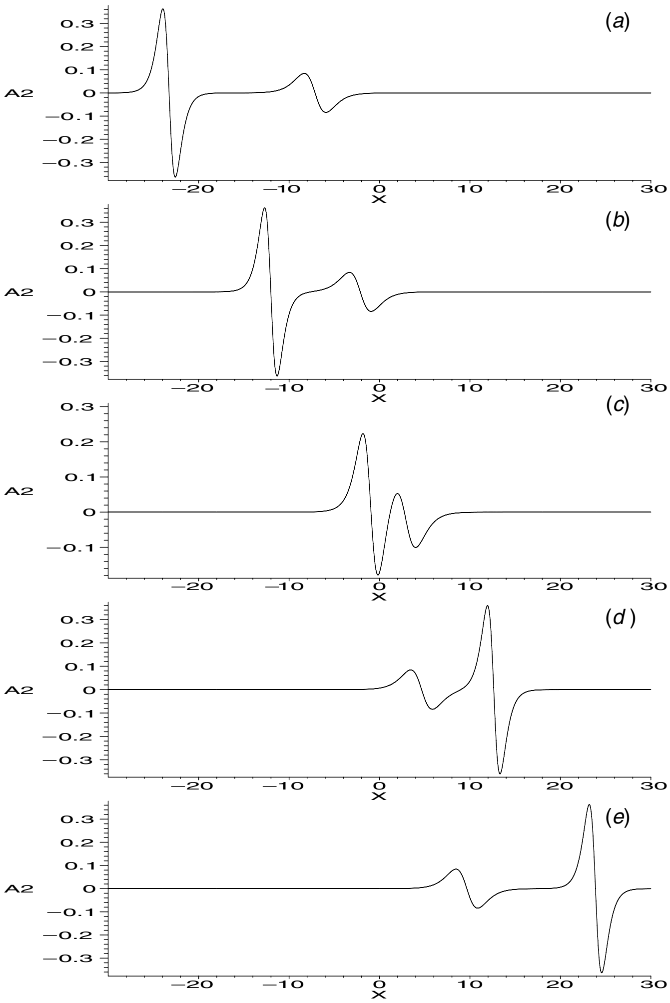

Figure 2. The interaction plots of the two-soliton solution for the field $A2 \equiv A_2$ expressed by (79) and (80) at times (*a*) $T = -10$, (*b*) $T = -5$, (*c*) $T = 0$, (*d*) $T = 5$ and (*e*) $T = 10$, respectively.

It is found that the coupled nonlinear system may possess many more abundant solution structures. This phenomenon has been observed before for the coupled non-integrable high-dimensional Klein–Gordon equation [14]. In this paper, we discover that when some kinds of model parameter conditions are satisfied, there may be some different types of τ-function and *multiple* soliton solutions.

The dynamics of atmospheric blockings has been one of the central and important problems since they are the main representations of the general circulation anormaly in the areas of mid-high latitudes. Atmospheric blocking events have a strong influence on regional weather and climate. The observations have shown that atmospheric blockings may locate in the mid-high latitudes, usually over the ocean, as the dipole pattern which was first discovered by Rex [15].

For the one-layer atmospheric model, the single soliton solution of the constant coefficient KdV equation is responsible for the dipole pattern of the atmospheric blockings. To explain the blocking life cycle, one has to use the soliton solutions of the variable coefficient KdV equation [16]. Using the similar analysis as the single-layer atmospheric model, the soliton solutions of the coupled KdV equation can also be utilized to explain the blockings under the two-layer atmospheric description. Similarly, the soliton solutions of the constant coefficient coupled KdV equation presented here cannot explain the blocking life cycle. To describe the blocking life cycle, one also has to extend the coupled KdV system (1)–(2) to the variable coefficient case. This problem will be studied in detail in the near future.

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